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Wound and rotating strings in $AdS_5 \times S^5$

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Abstract: From the Nambu-Goto string action using a non-diagonal gauge we construct a solution describing a wound string in $AdS_5 \times S^5$ which rotates in the same angular direction as the winding direction in S^5. The relation between energy and angular momentum of the string solution is characterized by the winding number and the bending number, and becomes linear in the large angular momentum limit. The small angular momentum limit is compared with the strong coupling limit of the integrable hamiltonian constructed in hep-th/0411089 by using the non-diagonal uniform gauge for the Polyakov string action. We analyze a wound string solution which is rotating in the different angular direction from the winding direction.

Keywords: AdS-CFT and dS-CFT Correspondence, Penrose limit and pp-wave background, Bosonic Strings.

Contents

1. Introduction

The AdS/CFT correspondence [1] has more and more revealed the deep relations between the weak coupling gauge theory and the strong coupling string theory, and vice versa. The obstacle to verify this conjecture beyond the supergravity approximation is the difficulty of quantizing the superstring theory in the $AdS_5 \times S^5$ background. However, the solvability of the string theory in the pp-wave background [2] has presented an important base to give an interesting proposal that the energies of specific free string excited states can be matched with the perturbative scaling dimensions of gauge invariant near-BPS operators with large R-charge in the BMN limit for the $\mathcal{N} = 4$ SU(N) super Yang-Mills (SYM) theory [3]. The BMN result has been interpreted as the semiclassical quantization of nearly point-like string with large angular momentum along the central circle of S^5 [4]. Various semiclassical extended string configurations with several large angular momenta in $AdS_5 \times S^5$ which usually go beyond the BMN scaling have been constructed extensively to study the AdS/CFT correspondence for non-BPS states [5 − 7] and reviewed in [8].

There has been an important step that the dilatation operator for the SO(6) sector in the planar $\mathcal{N} = 4$ SYM theory can be interpreted as a hamiltonian of an integrable spin chain in the one-loop approximation [9]. This observation has been generalized to studies of the complete dilatation operator [10] and the higher loop integrability [11](see [12] for review). The anomalous dimension of gauge invariant composite operator has been computed by using the Bethe ansatz for diagonalization of the dilatation operator. The one-loop Bethe ansatz in the SU(2) sector has been shown to match with the prediction of the one-loop semiclassical string approach [13]. Using the higher-loop Inozemtsev-Bethe ansatz, the matching has been shown to hold at two loops [14]. At the one-loop and two-loop orders there has been a general proof of the equivalence between the Bethe equation for the spin chain in the SU(2) sector and the classical Bethe equation for the classical $AdS_5 \times S^5$ string sigma model [15], whose approach has been further extended to the other sectors such as SL(2), SO(6) and so on [16]. The matching between the anomalous dimensions of gauge operators and the energies of dual semiclassical strings has been further

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confirmed up to two loops by constructing the conserved commuting charges and extending the other sectors [17, 18]. The other relevant aspects of the gauge/string duality have been investigated in [19]. The gauge/string duality has been also presented at the level of equations of motion [20], and at the level of effective action [21] where an interpolating spin chain sigma model action describing the continuum limit of the spin chain in the coherent basis is constructed. The approach based on the spin chain sigma model has been further investigated [22 − 24].

However, the matching has broken down at three loops [14, 17]. A similar three-loop disagreement has also appeared [25] in the string analysis of the $1/J$ quantum corrections to the near BMN states [26] where J is a single S^5 angular momentum. A fully reliable comparison requires the complete summation of the gauge theoretic perturbative expansion, that is, the spectrum of all-loop dilatation operator. In the closed SU(2) sector the three-loop dilatation operator coincides with the hamiltonian of the integrable Inozemtsev long-range spin chain [14], which however violates the BMN scaling at four loops. A new long-range spin chain which is different from the Inozemtsev long-range spin chain at four loops has been presented [27], where the all-loop asymptotic Bethe ansatz is proposed to recover the string-theoretic BMN formula. This all-loop ansatz has been further developed [28, 29] by adding the factorized scattering terms for the local excitations to make a particular discretization of the classical continuous Bethe equation for classical strings in [15]. This quantum Bethe ansatz [28] reprodeces the $1/J$ quantum corrections [25] to the energies of BMN string states and recovers the $\lambda^{1/4}$ asymptotics [4] of anomalous dimensions in the strong limit of the 't Hooft coupling constant λ, that are entirely determined by the novel scattering terms. This ansatz has a corresponding perturbative spin chain hamiltonian at weak coupling [30].

In ref. [31] from the phase-space approach to the Polyakov action for the bosonic closed string wound around an angular direction ϕ in S^5 and moving in $AdS_5 \times S^5$, the hamiltonian has been constructed to be of a square-root form and expressed in terms of an angular momentum J associated with the ϕ direction and the string tension $\sqrt{\lambda}$. The physical hamiltonian is derived by choosing the non-diagonal uniform gauge which is related to the static gauge by a 2d duality transformation [22]. In the sector of strings which do not wind around the ϕ direction the large J expansion of the physical hamiltonian with the BMN coupling $\lambda' = \lambda/J^2$ fixed reproduces the plane-wave hamiltonian and the $1/J$ and $1/J^2$ corrections in [25, 32] and in the strong coupling limit the $\lambda^{1/4}$ behavior is recovered. In the sector of strings which wind M times around the ϕ direction the string energy scales as $M\sqrt{\lambda}$ in the strong coupling limit. The gauge-fixed physical hamiltonian has been shown to be integrable by constructing the corresponding Lax representation. The integrability properties of superstring in $AdS_5 \times S^5$ have been further explored to investigate the Lax connections, their monodromies and the corresponding conserved charges [33]. On the other hand there have been various studies of comparing the quantum corrections to spinning string solutions in $AdS_5 \times S^5$ with the finite size corrections to the Bethe equations [34].

A non-diagonal gauge has been used for the Nambu-Goto action of the bosonic closed string moving in AdS_5 to construct a new type of string solution which is rotating and wound one time around an angular direction of AdS_3 embedded in AdS_5, and is stretched along the radial direction of AdS_3 with cusps or spikes [35]. In the large angular momentum limit the energy of string has been shown to be equal to the angular momentum with the sub-leading logarithmic anomalous term, where the motion of spikes is described by an effective classical mechanics that agrees with a coherent state description of the operators in the spin chain.

In order to understand the role of the winding number M in the integrable hamiltonian obtained by using the non-diagonal uniform gauge in [31], we will choose a non-diagonal gauge for the Nambu-Goto string action to construct a solution describing the string which stays at the origin of AdS_5 and is rotating and wound M times in the ϕ direction and stretched along an angular direction in S^5. The other stretched string solution wound in the ϕ direction but rotating in the other angular direction in S^5 will be constructed. In certain limits specified by the large angular momentum or the small one, the relations between the energy and the relevant angular momentum will be extracted for both string solutions.

2. Wound and rotating string solution with ω

Based on the Nambu-Goto string action we study a closed string located at the center of AdS_5 and rotating in S^5. The metric of $AdS_5 \times S^5$ is given by

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + \cos^2\gamma d\phi^2 + d\gamma^2 + \sin^2\gamma d\tilde{\Omega}_3^2 \,, \qquad (2.1)$$

where the radius of each subspace is unit and $d\Omega_3^2$, $d\tilde{\Omega}_3^2 = d\psi^2 + \cos^2\psi d\varphi_1^2 + \sin^2\psi d\varphi_2^2$ are metrics of separate three-spheres. We choose world-sheet coordinates in such a way that

$$t = \tau, \qquad \phi = \omega\tau + M\sigma \,, \qquad (2.2)$$

where M is the winding number. In order to express a configuration that a closed string is rotating in the φ_1 direction as well as the ϕ direction and stretched along the angular coordinate γ, we make further the following ansatz

$$\varphi_1 = \omega_1\tau \,, \qquad \gamma(\sigma) = \gamma(\sigma + 2\pi) \,, \qquad \psi = \varphi_2 = 0 \,. \qquad (2.3)$$

In this wound and rotating string configuration with $M \neq 0$ we note $\partial_\tau X^\mu \partial_\sigma X^\nu G_{\mu\nu} \neq 0$ and therefore the gauge choice (2.2) is not the diagonal conformal gauge. The non-diagonal gauge choice (2.2) with an AdS_5 angular coordinate ϕ was taken to construct a wound and rotating string with spikes in AdS_5 in ref. [35]. From the action

$$\begin{aligned} S_{NG} &= -\frac{\sqrt{\lambda}}{2\pi}\int d\tau d\sigma \sqrt{D} \,, \\ D &= -\gamma'^2(-1 + \sin^2\gamma\dot{\varphi}_1^2 + \cos^2\gamma\dot{\phi}^2) + \phi'^2\cos^2\gamma(1 - \sin^2\gamma\dot{\varphi}_1^2) \end{aligned} \qquad (2.4)$$

the equation of motion for ϕ yields

$$\partial_\sigma\left(\frac{(1 - \omega_1^2\sin^2\gamma)\cos^2\gamma}{\sqrt{D}}\right) = 0 \,, \qquad (2.5)$$

while that for φ_1 is trivially satisfied. Therefore we have a relation expressed in terms of an integration constant k as

$$\sqrt{D} = k(1-\omega_1^2\sin^2\gamma)\cos^2\gamma\,, \tag{2.6}$$

which combines with (2.4) to give

$$\gamma' = \cos\gamma\frac{\sqrt{(1-\omega_1^2\sin^2\gamma)[k^2\cos^2\gamma(1-\omega_1^2\sin^2\gamma)-M^2]}}{\sqrt{1-\omega_1^2\sin^2\gamma-\omega^2\cos^2\gamma}}. \tag{2.7}$$

The equation of motion for γ is written by

$$\begin{aligned}&\partial_\sigma\left(\frac{\gamma'}{\sqrt{D}}(1-\omega_1^2\sin^2\gamma-\omega^2\cos^2\gamma)\right) = \\ &\qquad = \frac{1}{\sqrt{D}}\sin\gamma\cos\gamma\left((\omega^2-\omega_1^2)\gamma'^2 - M^2(1+\omega_1^2\cos^2\gamma-\omega_1^2\sin^2\gamma)\right).\end{aligned} \tag{2.8}$$

Substituting (2.6) and (2.7) into (2.8) we can show that the involved equation (2.8) is indeed satisfied. Thus we obtain the first integral of the equation of motion.

Let us concentrate on a simpler case specified by $\omega \neq 0, \omega_1 = 0$. From the expression of $\gamma' = \cos\gamma\sqrt{k^2\cos^2\gamma - M^2}/\sqrt{1-\omega^2\cos^2\gamma}$ we have two restrictions $M/k \leq 1/\omega$ and $M/k \leq 1$. Therefore there are two possible conditions $0 < M/k \leq 1/\omega \leq 1$ and $0 < M/k \leq 1 \leq 1/\omega$. We first consider the former parameter condition where there appears a range which is bounded by two turning points as $M/k \leq \cos\gamma \leq 1/\omega$. Then $\gamma(\sigma)$ which is the length of a circular arc from the equator on the unit-radius S^2 parametrized by γ and ϕ, varies as σ increases from a minimum value $\gamma_- = \arccos(1/\omega)$ to a maximum vale $\gamma_+ = \arccos(M/k)$. At $\gamma = \gamma_-$, γ' diverges indicating the presence of a sharp wedge in the trajectory of $\gamma(\sigma)$ and at $\gamma = \gamma_+$, γ' vanishes so that there is a smooth peak between wedges. By gluing $2N$ of the arc segments we construct a string configuration wound M times around the ϕ direction with N wedges. The angle difference between the wedge and the peak is given by

$$\Delta\phi \equiv \frac{2\pi M}{2N} = \frac{\cos\gamma_+}{\cos\gamma_-}\int_{\gamma_-}^{\gamma_+} d\gamma\frac{1}{\cos\gamma}\sqrt{\frac{\cos^2\gamma_- - \cos^2\gamma}{\cos^2\gamma - \cos^2\gamma_+}}\,, \tag{2.9}$$

while the angular momentum J associated with the rotation in the ϕ direction and the energy E of the string are obtained by

$$J = \frac{\sqrt{\lambda}}{2\pi}2N\int_{\gamma_-}^{\gamma_+} d\gamma\cos\gamma\sqrt{\frac{\cos^2\gamma - \cos^2\gamma_+}{\cos^2\gamma_- - \cos^2\gamma}}\,, \tag{2.10}$$

$$E - \omega J = \frac{\sqrt{\lambda}N\omega}{\pi}\int_{\gamma_-}^{\gamma_+} d\gamma\cos\gamma\sqrt{\frac{\cos^2\gamma_- - \cos^2\gamma}{\cos^2\gamma - \cos^2\gamma_+}}\,. \tag{2.11}$$

These integrals are expressed in terms of complete elliptic integrals as

$$\Delta\phi = \frac{x_-^2}{x_+}\sqrt{\frac{1-x_+^2}{1-x_-^2}}\left[\Pi\left(\frac{q^2}{1-x_-^2}, q\right) - K(q)\right], \tag{2.12}$$

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$$J = \frac{\sqrt{\lambda}N}{\pi} x_+ [K(q) - E(q)] = \frac{\sqrt{\lambda}N}{4} x_+ q^2 F\left(\frac{3}{2}, \frac{1}{2}, 2, q^2\right), \tag{2.13}$$

$$E - \omega J = \frac{\sqrt{\lambda}N}{\pi} \frac{1}{\sqrt{1 - x_-^2}} \left[x_+ E(q) - \frac{x_-^2}{x_+} K(q) \right], \tag{2.14}$$

where $x_\pm = \sin\gamma_\pm$, $q = \sqrt{x_+^2 - x_-^2}/x_+$ and $F(\frac{3}{2}, \frac{1}{2}, 2, q^2)$ is the hypergeometric function. Thus some combinations of the integration constants such as k and ω appear as parameters of the elliptic functions. From (2.13) and (2.14) the energy expression is extracted as

$$E = \frac{\sqrt{\lambda}N}{\pi} \frac{x_+}{\sqrt{1 - x_-^2}} q^2 K(q) . \tag{2.15}$$

Eliminating x_+ and x_- in (2.12), (2.13) and (2.15) we determine the energy E as a function of the angular momentum J, the winding number M and the bending number N.

We will derive explicit analytic expressions in three different regions. From (2.13) and (2.15) the region of large E and large J as compared to $\sqrt{\lambda}$ corresponds to a region $q \approx 1$ which means that $x_- \approx 0$, that is, ω approaches 1 from above. Thus the large J region is specified by $r_- \approx 0$. Both E and J diverge logarithmically owing to $K(q)$ and are expressed to the leading order as

$$E \approx J \approx \frac{\sqrt{\lambda}N}{\pi} x_+ \log \frac{x_+}{x_-}, \tag{2.16}$$

while the difference $E - J$ in (2.14) becomes finite

$$E - J \approx E - \omega J \approx \frac{\sqrt{\lambda}N}{\pi} x_+ \tag{2.17}$$

owing to the factor x_-^2 which suppresses the logarithmic divergence of $K(q)$. Here we write down a formula

$$\Pi(n, q) = K(q) + \frac{2\sqrt{1 - k'^2 \sin^2\psi}}{k'^2 \sin 2\psi} \left[F(\psi, k')K(q) - E(\psi, k')K(q) - F(\psi, k')E(q) + \frac{\pi}{2} \right] \tag{2.18}$$

with $k'^2 = 1 - q^2$, $n = 1 - k'^2 \sin^2\psi$. For $n \approx q^2 \approx 1$ it approximately reduces to

$$\Pi(n, q) \approx \sqrt{\frac{n}{(1-n)(n-q^2)}} \left(\frac{\pi}{2} - \arcsin\sqrt{\frac{1-n}{1-q^2}} \right), \tag{2.19}$$

whose substitution into (2.12) provides

$$\cos\Delta\phi \approx \sqrt{1 - x_+^2}, \tag{2.20}$$

which implies $2M \le N$. Thus x_+ is characterized by M and N as

$$x_+ \approx \sin\frac{M\pi}{N}. \tag{2.21}$$

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From (2.17) and (2.21) we get

$$E - J \approx \frac{\sqrt{\lambda}N}{\pi} \sin \frac{M\pi}{N} \tag{2.22}$$

in the large angular momentum limit. For the special $2M = N$ case that corresponds to $\gamma_+ = \pi/2$, it reduces to

$$E - J \approx \frac{\sqrt{\lambda}N}{\pi}, \tag{2.23}$$

which, for $N = 2$ with $M = 1$ shows the same expression as that in [4]. In the general situations there is a difference that our closed string configuration is a zigzag bending curve surrounding the north pole on the relevant $S^2(\gamma, \phi)$, while the closed string presented in [4] is folded on itself along the meridian in a line form. For instance in the $M = 1$ and $N = 3$ case the string shape at fixed time starts at the equator point specified by $\gamma = \phi = 0$ that is one wedge to reach the maximum peak point $\gamma = \pi/3$ with $\phi = \pi/3$ and turns around to the second wedge $\gamma = 0,\ \phi = 2\pi/3$. This one step is repeated three times to arrive at the starting point. In the special $M = 1$ and $N = 2$ case the string shape starts at the same equator point and goes along the meridian to reach the north pole and turns around to $\gamma = 0,\ \phi = \pi$. Then it returns back on itself so that this configuration just shows the string folded in the γ direction and composed of the four segments. Therefore this picture is consistent with the reconstruction of the energy-spin relation for the four folded string as a special case from the general relation (2.22). Similarly for the $2M = N$ case the string is folded $2N$ times in the γ direction.

In view of (2.13) and (2.15) the small J and small E region is specified by small q. The two parameters x_+, x_- are traded for m, n as follows

$$m = q^2 = \frac{x_+^2 - x_-^2}{x_+^2}, \qquad n = \frac{q^2}{1 - x_-^2} = \frac{x_+^2 - x_-^2}{x_+^2(1 - x_-^2)}, \tag{2.24}$$

which satisfy $0 < m \le n < 1$ and inversely give $x_-^2 = (n-m)/n,\ x_+^2 = (n-m)/(n(1-m))$. The energy, the angular momentum and the angle difference are expressed in terms of m, n as

$$E = \frac{\sqrt{\lambda}N}{2}\sqrt{\frac{m(n-m)}{1-m}}\left(1 + \frac{m}{4} + \cdots\right), \tag{2.25}$$

$$J = \frac{\sqrt{\lambda}N}{4} m \sqrt{\frac{n-m}{n(1-m)}}\left(1 + \frac{3}{8}m + \cdots\right), \tag{2.26}$$

$$\Delta\phi = F(\psi, k')K(q) - E(\psi, k')K(q) - F(\psi, k')E(q) + \frac{\pi}{2}, \tag{2.27}$$

where $q = \sqrt{m}, k' = 1 - m$ and

$$\sin^2 \psi = \frac{1-n}{1-m}. \tag{2.28}$$

There are two different small q regions: one region is specified by $m/n \ll 1$ and $m \ll 1$, and the other one by $n \approx m \ll 1 - m/n \ll 1$.

Let us consider the parameter region specified by $m \ll n \ll 1$ that is more restricted than the former one $m \ll n < 1$. Since this parameter region means $x_+ \approx x_- \approx 1$, that is, $\gamma_+ \approx \gamma_- \approx \pi/2$, the rotating and bending string is so located near the north pole that it is of a small size. From (2.25) and (2.26) we make the leading estimations such as $E \approx \sqrt{\lambda} N \sqrt{mn}/2$ and $J \approx \sqrt{\lambda} N m/4$ so that small E and small J are characterized by small m. The elimination of m yields

$$E^2 \approx Nn\sqrt{\lambda} J \,, \tag{2.29}$$

which however, includes a parameter n, that will be expressed in terms of J, M and N through (2.27). The relation (2.28) gives $\psi \approx \pi/2$ to the leading order, from which the $\Delta\phi$ in (2.27) becomes $K(k')K(q) - E(k')K(q) - K(k')E(q) + \pi/2$, that is identically zero. The expansion of (2.28) around $\psi = \pi/2$ gives here for $n \ll 1,\ m \ll 1$

$$(\Delta\psi)^2 \approx n - m + \frac{1}{3}(n^2 + mn - 2m^2) \,, \qquad \psi \equiv \frac{\pi}{2} - \Delta\psi \tag{2.30}$$

and the expansion of (2.27) yields

$$\Delta\phi \approx \frac{\pi}{4} q \Delta\psi \,, \tag{2.31}$$

where we have used

$$\begin{aligned} \partial_\psi F(\psi, k') &= (1 - k'^2 \sin^2\psi)^{-1/2} \,, \qquad \partial_\psi^2 F(\psi, k')|_{\psi=\pi/2} = 0 \,, \\ \partial_\psi E(\psi, k') &= (1 - k'^2 \sin^2\psi)^{1/2} \,, \qquad \partial_\psi^2 E(\psi, k')|_{\psi=\pi/2} = 0 \,. \end{aligned} \tag{2.32}$$

The leading value $n \approx (4M/N)^2/m \approx 4M^2\sqrt{\lambda}/JN$ determined from (2.31) makes (2.29) change into $E^2 \approx 4M^2\lambda$. By estimating the sub-leading term from (2.31), (2.25) and (2.26) we obtain

$$E^2 \approx 4M^2 \left(\lambda + \frac{14}{3} \frac{J}{N} \sqrt{\lambda} \right) , \tag{2.33}$$

which holds in the regions $J \ll E$ and $M/N \approx 0$. We note that the first term corresponds to the $M\sqrt{\lambda}$ term which is determined from the integrable hamiltonian in the strong coupling limit for $M \neq 0$ [31].

Here we consider the other parameter region $n \approx m \ll 1 - m/n \ll 1$ which implies $x_+ \approx x_- \ll 1$, that is, $\gamma_+ \approx \gamma_- \approx 0$. Therefore the rotating string is located near the equator and wound nearly around a central circle of S^5 so that it is not of a small size although J is small. For this particular parameter region, the eqs. (2.25) and (2.26) yield a linear relation, to the leading order

$$E \approx 2J \,, \tag{2.34}$$

which holds in the region $M/N \approx 0$. The leading expression of (2.30) combines with (2.31) and (2.26) to make (2.34) change into $E \approx M\sqrt{\lambda}$.

Now we analyze the other parameter condition $0 < M/k \leq 1 \leq 1/\omega$. Since there is a range $M/k \leq \cos\gamma \leq 1$ where γ' vanishes at $\gamma = \gamma_+ \equiv \arccos(M/k)$ and γ' takes $\sqrt{(k^2 - M^2)/(1-\omega^2)}$ at $\gamma = 0$, the expressions in (2.9), (2.10), (2.11) are changed such that the integrations are taken from 0 to γ_+ and $\cos\gamma_-$ is replaced by $1/\omega$. Then we have

$$\Delta\phi = \omega x_-^2 \sqrt{\frac{1-x_+^2}{x_+^2 + x_-^2}} \Pi\left((1+x_-^2)q^2, q\right), \tag{2.35}$$

$$J = \frac{\sqrt{\lambda}N}{\pi} \sqrt{x_+^2 + x_-^2}[K(q) - E(q)], \tag{2.36}$$

$$E - \omega J = \frac{\sqrt{\lambda}N}{\pi} \omega \sqrt{x_+^2 + x_-^2} E(q), \tag{2.37}$$

$$E = \frac{\sqrt{\lambda}N}{\pi} \sqrt{\frac{x_+^2 + x_-^2}{1+x_-^2}} K(q), \tag{2.38}$$

where $x_+ = \sin\gamma_+, x_-^2 = 1/\omega^2 - 1$ and $q = x_+/\sqrt{x_+^2 + x_-^2}$. The large E and large J region is specified by $q \approx 1$, that is, $x_- \approx 0$ where ω approaches 1 from below. Using (2.35) and (2.37) we again derive the energy-spin relation (2.22) together with (2.21). Thus we see that this solution for $0 < M/k \leq 1 \leq 1/\omega$ in the large J limit is smoothly connected to the above argued solution for $0 < M/k \leq 1/\omega \leq 1$. In terms of $m = q^2, n = x_+^2(1 + x_-^2)/(x_+^2 + x_-^2)$ the energy, the angular momentum and the angle difference are expressed as

$$E = \frac{\sqrt{\lambda}N}{2} \sqrt{\frac{(n-m)}{n(1-m)}} \left(1 + \frac{m}{4} + \cdots\right), \tag{2.39}$$

$$J = \frac{\sqrt{\lambda}N}{4} \sqrt{\frac{m(n-m)}{1-m}} \left(1 + \frac{3}{8}m + \cdots\right), \tag{2.40}$$

$$\Delta\phi = \sqrt{\frac{(1-n)(n-m)}{n}} \Pi(n, \sqrt{m}), \tag{2.41}$$

where $0 < m \leq n < 1$ and the small J is specified by small m. In the more restricted parameter region $m \ll n^2 \ll n \ll 1$ that corresponds to $\gamma_+ \approx 0, \omega \approx 0$ the energy-spin relation for very small J is estimated as

$$E \approx M\sqrt{\lambda} + \frac{3}{1 - \left(\frac{2M}{N}\right)^2} \frac{J^2}{M\sqrt{\lambda}}, \tag{2.42}$$

which holds in the regions $J \ll E$ and $M/N \approx 1/2$ with $J/M\sqrt{1-(2M/N)^2} \ll \sqrt{\lambda}$, while in the other parameter region $n \approx m \ll 1 - m/n \ll 1$ that corresponds to $\gamma_+ \approx 0, \omega \approx 1$ it is given by

$$E \approx M\sqrt{\lambda} + \frac{3}{2}\sqrt{1 - \left(\frac{2M}{N}\right)^2} J, \tag{2.43}$$

which holds in the region $M/N \approx 0$. For the latter region the critical limit $\omega \to 1$ from below means $x_- = \sqrt{1/\omega^2 - 1} \to 0$, that is, $n - m \to 0$ so that the energy E as well as the angular momentum J reduces to zero. This degenerate limit is connected the critical limit $\omega \to 1$ from above in (2.34) which is taken by $x_- = \sqrt{1 - 1/\omega^2} \to 0$, that is, $n - m \to 0$ where both E and J also vanish.

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3. Wound and rotating string solution with ω_1

We turn to the other simpler case specified by $\omega_1 \neq 0$, $\omega = 0$. In this case the gauge choice (2.2) belongs to the diagonal conformal gauge. The first integral (2.7) is expressed as

$$\gamma' = \cos\gamma\sqrt{k^2\cos^2\gamma(1-\omega_1^2\sin^2\gamma)-M^2} = k\omega_1\cos\gamma\sqrt{(\cos^2\gamma-\alpha_+)(\cos^2\gamma+\alpha_-)}\,,$$

$$\alpha_\pm = \frac{\sqrt{(\omega_1^2-1)^2+4\omega_1^2M^2/k^2}\pm(\omega_1^2-1)}{2\omega_1^2}\,. \tag{3.1}$$

From this expresson we have one restriction $\alpha_+ \le 1$ which implies $M/k \le 1$, so that there are two possible conditions $0 < \alpha_- \le \alpha_+ \le 1$ and $0 < \alpha_+ \le 1$ with $\alpha_+ \le \alpha_-$. One condition is expressed as $M \le k$ with $1 \le \omega_1$ while the other one $M \le k$ with $\omega_1 \le 1$. We start to consider the former case. Then $\gamma(\sigma)$ on $S^2(\gamma,\phi)$ embedded in S^5, varies from zero where $\partial_\phi\gamma$ takes $\sqrt{(k/M)^2-1}$ to a maximum value $\gamma_+ = \arccos\sqrt{\alpha_+}$ where $\partial_\phi\gamma$ vanishes, and $\gamma(\sigma)$ returns back to zero where $\partial_\phi\gamma = -\sqrt{(k/M)^2-1}$. The $2N$ gluing of the arc segments yields a string configuration wound M times around the ϕ direction with N wedges. The acute angle of a wedge is locally chraracterized by $2(\pi/2 - \arctan\sqrt{(k/M)^2-1})$. From the view point of the string configuration on the other embedded two-sphere $S^2(\gamma,\varphi_1)$ parametrized by γ and φ_1, $\gamma(\sigma)$ which is the length of circular arc from the north pole on it varies from zero to a maximum value γ_+ and returns back on itself. The angle difference between the wedge and the peak on $S^2(\gamma,\phi)$ is expressed as

$$\Delta\phi \equiv \frac{2\pi M}{2N} = \frac{M}{k\omega_1}\int_0^{\gamma_+} d\gamma \frac{1}{\cos\gamma\sqrt{(\cos^2\gamma-\alpha_+)(\cos^2\gamma+\alpha_-)}}\,, \tag{3.2}$$

while the angular momentum coming from the rotation in the φ_1 direction and the energy of the string are described by

$$J_1 = \frac{\sqrt{\lambda}}{2\pi}2N\int_0^{\gamma_+} d\gamma \frac{\sin^2\gamma\cos\gamma}{\sqrt{(\cos^2\gamma-\alpha_+)(\cos^2\gamma+\alpha_-)}}\,, \tag{3.3}$$

$$E - \omega_1 J_1 = \frac{\sqrt{\lambda}N}{\pi\omega_1}\int_0^{\gamma_+} d\gamma \frac{\cos\gamma(1-\omega_1^2\sin^2\gamma)}{\sqrt{(\cos^2\gamma-\alpha_+)(\cos^2\gamma+\alpha_-)}}\,. \tag{3.4}$$

From (3.1) the two parameters ω_1, k are traded for α_+, α_- so that ω_1, k in (3.2) and (3.4) are implicitly expressed in terms of α_+, α_-. The integrations in (3.2) and (3.3) are performed as

$$\Delta\phi = \frac{M}{k\omega_1}\frac{1}{\sqrt{1+\alpha_-}}\Pi(1-\alpha_+, q)\,, \tag{3.5}$$

$$J_1 = \frac{\sqrt{\lambda}N}{\pi}\sqrt{1+\alpha_-}(K(q)-E(q)) = \frac{\sqrt{\lambda}N}{4}\sqrt{1+\alpha_-}q^2 F\left(\frac{3}{2},\frac{1}{2},2,q^2\right)\,, \tag{3.6}$$

where $q^2 = (1-\alpha_+)/(1+\alpha_-)$. A comparison of (3.3) with (3.4) yields

$$E = \frac{\sqrt{\lambda}N}{\pi\omega_1}\frac{1}{\sqrt{1+\alpha_-}}K(q)\,. \tag{3.7}$$

The eqs. (3.6) and (3.7) show the similar expressions to J in (2.13) and E in (2.15) respectively, although the string configurations are different. The difference $E - \omega_1 J_1$ in (3.4) is then provided by

$$E - \omega_1 J_1 = \frac{\sqrt{\lambda} N}{\pi \omega_1} \frac{1}{\sqrt{1+\alpha_-}} [\omega_1^2 (1+\alpha_-) E(q) + (1 - \omega_1^2(1+\alpha_-)) K(q)] \,, \tag{3.8}$$

which resembles (2.14).

From (3.6) and (3.7) large J_1 and large E are achieved by taking q to be near the critical value 1. This region is specified by turning α_+ and α_- close to zero, which corresponds to $M/k \approx 0$, $\omega_1 \approx 1$. Because of the suppression factor $(1 - \omega_1^2(1+\alpha_-))$ multiplying $K(q)$ in (3.8) the difference $E - J_1$ becomes finite

$$E - J_1 \approx E - \omega_1 J_1 \approx \frac{\sqrt{\lambda} N}{\pi} \,, \tag{3.9}$$

which shows the same expression as (2.23). In the small $\alpha_\pm$ region, $0 \approx \alpha_- \le \alpha_+$ where $\alpha_+ \approx 0$ implies $\gamma_+ \approx \pi/2$ the use of (2.19) for (3.5) gives

$$\cos \Delta\phi \approx \sqrt{\frac{\alpha_+(1+\alpha_-)}{\alpha_+ + \alpha_-}} \,, \tag{3.10}$$

which also implies $2M \le N$. When ω_1 approaches 1 from above we have $\cos\Delta\phi \approx 1$, that is, $M/N \ll 1$. In the special $M = 0$ case the string is not wound in the ϕ direction but folded $2N$ times in the γ direction. When $N = 2$ with $M = 0$ the energy-spin relation (3.9) again coincides with that for the folded string composed of the four segments in ref. [4]. Alternatively if we substitute $M = 0$ directly into (3.6) and (3.7) we have

$$J_1 = \frac{\sqrt{\lambda} N}{\pi} (K(q) - E(q)) \,, \qquad E = \frac{\sqrt{\lambda} N}{\pi \omega_1} K(q) \tag{3.11}$$

with $q = 1/\omega_1$, which, for $N = 2$, reduces to those in ref. [4].

We consider the opposite parameter region where J_1 is small. From (3.6) it is specified by small q, that is, $\alpha_+ \approx 1$, which leads to the lower bound $k \approx M$ through (3.1). Because of $\gamma_+ \approx 0$ the rotating string is wound nearly around the equator. We expand $\alpha_\pm$ in (3.1) as $\alpha_+ = 1 - (1+\alpha_-) q^2 \approx 1 - \delta/(\omega_1^2 + 1)$, $\alpha_- \approx (1 - \omega_1^2 \delta/(\omega_1^2+1))/\omega_1^2$ with $\delta \equiv 1 - M^2/k^2$. Plugging the following expansion into (3.7)

$$q^2 \approx \frac{4 J_1}{\sqrt{\lambda} N \sqrt{1+\alpha_-}} - \frac{6 J_1^2}{\lambda N^2 (1+\alpha_-)} \,, \tag{3.12}$$

which is derived from (3.6), we have

$$E \approx \frac{\sqrt{\lambda} N}{2\omega_1 \sqrt{1+\alpha_-}} + \frac{J_1}{2\omega_1(1+\alpha_-)} \tag{3.13}$$

with $\alpha_- \approx (1 - \omega_1^2(1+\alpha_-) q^2)/\omega_1^2$. The remaining condition (3.5) is expressed as

$$\frac{M}{N} \approx \frac{1}{2\omega_1} \sqrt{\frac{1 - (\omega_1^2+1)(1+\alpha_-) q^2}{1+\alpha_-}} \left[1 + \left(\frac{1}{4} + \frac{\sqrt{1+\alpha_-}}{2} \right) q^2 \right] , \tag{3.14}$$

where the expansion of (2.18) around $\psi = \pi/2$ is used. Focusing on $NJ_1/M^2 \ll \sqrt{\lambda}$ and the large N/M region that corresponds to the large ω_1 region we can estimate the energy as

$$E \approx \sqrt{\lambda} M \left(1 + \frac{N}{2M^2} \frac{J_1}{\sqrt{\lambda}} \right), \tag{3.15}$$

which scales as $\sqrt{\lambda}M$ in the strong coupling limit. For comparison manipulating the equations in (3.11) we express the squared energy for the folded string with small angular momentum as

$$E^2 \approx \sqrt{\lambda} N \left(J_1 + \frac{1}{2} \frac{J_1^2}{\sqrt{\lambda}N} + \frac{1}{4} \frac{J_1^3}{(\sqrt{\lambda}N)^2} \right), \tag{3.16}$$

which shows the Regge behavior and the $\lambda^{1/4}$ scaling for energy in the strong coupling limit.

Here we consider the other parameter condition $M \leq k$ with $\omega_1 \leq 1$, where the angle difference, the angular momentum and the energy are also given by the same expressions as (3.5), (3.6) and (3.7). In the large J_1 and large E region that corresponds to the small $\alpha_\pm$ region, $0 \approx \alpha_+ \leq \alpha_-$. In this region where ω_1 approaches 1 from below and $\gamma_+ \approx \pi/2$, the linear energy-spin relation (3.9) also holds and the eq. (3.10) leads to $M/N \approx 1/2$. In the large angular momentum limit specified by $2M = N$ the string is again folded in the γ direction and composed of the $2N$ segments. In the small J_1 region which is specified by $\alpha_+ \approx 1$, that is, $\gamma_+ \approx 0$ we have derived the energy-spin relation (3.15) by focusing on the large ω_1 region that corresponds to $M/N \approx 0$, while here we can focus on the small ω_1 region that corrsponds to $M/N \approx 1/2$. In this particular region the energy-spin relation is estimated as

$$E \approx M\sqrt{\lambda} + \frac{3N}{4M} \sqrt{1 - \left(\frac{2M}{N} \right)^2} J_1 \,. \tag{3.17}$$

4. Conclusion

Solving the non-linear equations of motion derived from the Nambu-Goto action for the string moving in $AdS_5 \times S^5$ we have presented a solution describing a closed string rotating and wound M times in the same ϕ direction, as well as a solution describing a closed string wound in the ϕ direction but rotating in the other angular direction of S^5. The gauge choice for the former string configuration is not the diagonal conformal gauge, unlike the latter one. We have demonstrated that the string energies in certain limits are explicitly described in terms of the angular momentum J, the winding number M and the bending number N.

For the former configuration we have observed that in the large angular momentum limit one of the two turning points for the shape trajectory of the wound string is located at the equator of the relevant S^2, while the other turning point is specified by the ratio M/N. For the special $2M = N$ case the rotating string of the bending hoop configuration changes into that of the $2N$ folded line configuration. In the large angular momentum limit the energy of the wound and rotating string has been shown to be equal to the large angular momentum with the sub-leading term which is specified by the winding number M and the

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bending number N. On the other hand in the small angular momentum region we have seen that there exist three types of string configurations, a short string which is located near the north pole of the S^2 and characterized by $M/N \approx 0$, a string wound nearly along its equator with $M/N \approx 0$ and that with $M/N \approx 1/2$. We have demonstrated that the energy of these strings is described by the string tension times the winding number M in the small angular momentum limit, whose behavior is similar to the strong coupling limit of the integrable hamiltonian for $M \neq 0$ in ref. [31].

For the latter configuration the wound and rotating strings in the large angular momentum limit which are specified by $M = 0$ and $2M = N$, have been shown to become the $2N$ folded and rotating strings and yield the same linear energy-spin relation with the sub-leading term specified by the bending number N. In the small angular momentum region we have extracted the energy-spin relations for the rotating strings specified by the two parameter regions $M/N \approx 0$ and $M/N \approx 1/2$. The energy of these rotating strings wound nearly along the equator of the relevant S^2 has been also shown to be the string tension times the winding number M in the small angular momentum limit. It is desirable to investigate what gauge invariant composite operators correspond to the wound and rotating string solutions labelled with the winding and bending numbers.

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